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Some relations between twisted K-theory and E_8 gauge theory

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Abstract: Recently, Diaconescu, Moore and Witten provided a nontrivial link between K-theory and M-theory, by deriving the partition function of the Ramond-Ramond fields of type IIA string theory from an E_8 gauge theory in eleven dimensions. We give some relations between twisted K-theory and M-theory by adapting the method of [1, 2]. In particular, we construct the twisted K-theory torus which defines the partition function, and also discuss the problem from the loop group picture, in which the Dixmier-Douady class is the Neveu-Schwarz field. In the process of doing this, we encounter some mathematics that is new to the physics literature. In particular, the eta differential form, which is the generalization of the eta invariant, arises naturally in this context. We conclude with several open problems in mathematics and string theory.

Keywords: M-Theory, Anomalies in Field and String Theories.

Contents

1. Introduction and setup

Type II string theories in ten dimensions contain in addition to gravity and fermions, p-form fields, the Ramond-Ramond RR and the Neveu-Schwarz NS fields. D-branes are charged [3] under those p-forms. It is by now well known that RR charges in the absence of NS fields can be classified by K-theory of spacetime [4], namely by $K^0(X)$ for type IIB [5] and by $K^1(X)$ for type IIA [6]. The RR fields are also classified by K-theory [7, 8], with the roles of K^0 and K^1 interchanged. In the presence of a NS B-field, or its field strength H_3, the fields and the charges are classified by twisted K-theory, in the sense of [9], as was shown in [10, 11] by analysis of worldsheet anomalies for the case the NS field $[H_3] \in H^3(X, \mathbb{Z})$ is a torsion class, and in [12] for the nontorsion case.

M-theory is a theory in eleven dimensions which is not yet known except in specific regions (or points) of its "moduli space". It has been shown by Witten [13, 14] that the topological part can be encoded in the index theory of an E_8 gauge bundle. At the level of supergravity, the low energy limit of string theories and M-theory, there is an explicit relation between the two given by Kaluza-Klein reduction. The story is much more subtle at the quantum level due to the existence of nontrivial phase factors in the partition functions. In a rather nontrivial way, it has been recently shown by [1, 15] that one can also relate the corresponding partition functions, namely the one derived using the E_8 theory in eleven dimensions and the one derived from K-theory in ten dimensions. The authors restricted themselves mostly to the RR sector. This has been generalized in [2] to include the fermions, one-loop contributions and membrane instantons, as well as including flat background NS potentials. The authors show, nontrivially, that the partition functions are T-duality invariant only after including the above effects. They also identify the T-duality anomalies. In both [1, 15] and [2] (see also [16]), it has been suggested to include nontrivial $[H_3]$.

In this paper, we attempt at generalizing some of the ideas along the lines of [1, 15, 2] in the context of twisted K-theory. A convenient computational tool for twisted K-theory is the Atiyah-Hirzebruch spectral sequence (AHSS) [17], which has been nicely used in analyzing physical D-brane configurations (see e.g. [18, 19, 20]). Inspite of the apparent physical favor of AHSS, we prefer here to work in the full twisted K-theory.

The spaces we deal with are the following. Y is an eleven-dimensional spin manifold corresponding to M-theory. X is a ten-dimensional manifold which is the base for a circle bundle with total space Y, and corresponds to type-IIA superstring theory. Finally, Z is a twelve-dimensional manifold which is a disk bundle over X, whose boundary is the circle bundle Y over X.

The basic setup for the bundles we consider is given in the following diagram

$$\begin{array}{ccc} E_8 & \to & P \\ & & \downarrow \\ S^1 & \to & Y \\ & & \downarrow \pi \\ & & X \end{array} \tag{1.1}$$

where P is a principal E_8 bundle over the 11-dimensional manifold Y, which in turn is a principal S^1 bundle over the 10-dimensional manifold X. Then Y has a supergravity field whose field strength is a closed 4-form G_4, that is related to the integral characteristic class invariant a of P as follows:

$$\frac{G_4}{2\pi} = a - \frac{\lambda}{2}\,, \tag{1.2}$$

where $\lambda/2$ is equal to half the first Pontrjagin class $p_1(Y)$ of Y.

$$\begin{tikzcd} Y \arrow[r, "\text{large volume limit}"] \arrow[d, "\text{weak string coupling}"] & \text{classical M} - \text{theory} \arrow[d, leftrightarrow, "\text{comparison}"] \\ X \arrow[r, "\text{large volume limit}"] & \text{classical IIA} \end{tikzcd}$$

We are interested in the comparison, using the metric $g_Y = t\pi^*(g_X) + \pi^*(e^{2\phi/3})\mathcal{A}\otimes\mathcal{A}$, in the large volume, adiabatic limit as $t \to \infty$. We explain the notation in section 4.

This paper is organized as follows. In section 2 we review the relation between M-theory and E_8 gauge theory, and then the M-theory partition function in section 3, and write the phase in terms of the reduced eta invariant. In section 4 we relate the eleven dimensional fields to the ten dimensional ones by dimensional reduction, and relate the eta invariant in the adiabatic limit to an integral involving the eta form. In section 5 we study the twisted K-theory description of type IIA partition function and in particular the twisted K-theory theta functions. We relate the E_8 bundle on Y to an LE_8 bundle on X, which gives rise to the Neveu-Schwarz H_3 field (i.e. the twist), as well as a class in twisted K-theory (over the rationals). In section 6 we conclude with discussions and some open problems.

2. M-theory

M-theory [21]–[23] has three kinds of impurities: membranes, fivebranes and boundaries. The low energy theory is eleven-dimensional supergravity. The massless degrees of freedom are the metric g , a three-form potential C_3, and a Rarita-Schwinger fermionic spin 3/2 field ψ_M. The action of eleven dimensional supergravity is [24]

$$I_{11} = I_{\text{grav}} + I_{\text{G}_4} + I_{\text{C.S.}} + I_{\text{Fermi}} + I_{\text{coupling}} \tag{2.1}$$

where

$$I_{\text{grav}} = \frac{1}{2\kappa_{11}^2}\int_Y \hat{\mathcal{R}}\, d\text{vol} \tag{2.2}$$

$$I_{G_4} = -\frac{1}{2\kappa_{11}^2}\frac{1}{2\cdot 4!}\int_Y |G_4|^2\, d\text{vol} \tag{2.3}$$

$$I_{\text{C.S.}} = -\frac{1}{12\kappa_{11}^2}\int_Y C_3\wedge G_4\wedge G_4 \tag{2.4}$$

$$I_{\text{Fermi}} = \frac{1}{2\kappa_{11}^2}\frac{1}{2}\int_Y \bar{\psi} D_{\text{R.S.}}\psi\, d\text{vol}\,, \tag{2.5}$$

where $d\text{vol} = d^{11}x\sqrt{-g}$, $\hat{\mathcal{R}}$ is the scalar curvature of Y, G_4 is the four-form field strength, which, when cohomologically trivial, is equal to dC_3. The fermions involve the kinetic action of ψ_M involving the Rarita-Schwinger operator $D_{\text{R.S}}$. One can view $D_{\text{R.S.}}$ as [1, 15] the Dirac operator coupled to the vector bundle associated to the virtual bundle $TY - 3\mathcal{O}$, where the $\mathcal{O}$ factors correspond to subtraction of ghosts. I_{coupling} corresponds to coupling of ψ_M to G_4 as well as quartic ψ_M self-couplings. It is not essential for our discussion and thus we do not record it.

The source-free Bianchi identity and equation of motion are

$$dG_4 = 0 \tag{2.6}$$

$$d * G_4 = -\frac{1}{2}G_4\wedge G_4 \tag{2.7}$$

which can be modified by adding sources, namely the membrane $M2$ and the fivebrane $M5$, respectively. They have worldvolumes, respectively, $\mathcal{W}_3$ and $\mathcal{W}_6$ embedded by an embedding ι in spacetime Y. There are also one-loop corrections that, for example, modify the RHS of (2.7) by the topological quantity $X_8 = \frac{1}{192}(p_1^2 - 4p_2)$, given in terms of the Pontrjagin classes of the tangent bundle TY.

The four-form of M-theory obeys the quantization condition [13]

$$\frac{G_4}{2\pi} + w_4 \in H^4(Y;\mathbb{Z})\,, \tag{2.8}$$

where w_4 is the fourth Stiefel-Whitney class of the tangent bundle TY. In the orientable case, which is what we are interested in, $w_4 = \lambda/2 \bmod 1$, with $\lambda = p_1/2$.

3. M-theory partition function

This section is mostly a review of relevant facts. The M-theory partition function is a product of factors corresponding to the different parts of the action (2.1),

$$Z_M \sim Z_{\rm grav} Z_{G_4} Z_{\rm C.S.} Z_{\rm Fermi} Z_{\rm coupling} \tag{3.1}$$

We are interested in the topological part of the partition function, which means that,as in [1], we keep only the moduli associated with C_3 (or G_4) but keep all the phases, so that the part we are interested in is

$$e^{-||G_4(a)||^2}\Omega_M(C_3)\,. \tag{3.2}$$

Consequently, we do not consider $Z_{\rm grav}$ nor $Z_{\rm coupling}$. This theory can be viewed as having two kinds of fermions. First, the spin 1/2 fermions in the E_8 gauge theory,[1] and then the spin 3/2 Rarita-Schwinger fields in the supergravity. At the level of actions, we have

$$I_M = I_{E_8} + \frac{I_{\rm R.S.}}{2} \tag{3.3}$$

The low energy quantum measure of M-theory factorizes in terms of manifestly well-defined factors[2]

$$\det D_{\rm R.S.} e^{iI_M} = \left\{\det D_{\rm R.S.} e^{iI_{\rm R.S.}/2}\right\} \cdot e^{iI_{E_8}}\,. \tag{3.4}$$

The expression $G_4 = dC_3$ is not valid globally and C_3 is not a well-defined differential form, implies that one has to be careful in defining the topological part $I_{\rm C.S.}$ of the action I_{11}.

The way around this is to lift to twelve dimensions and look at the action

$$I_{12} \sim \int_Z G_4 \wedge G_4 \wedge G_4\,. \tag{3.5}$$

[1]Note added in the proof: we do not address whether or not those are physical fermions. Some analysis on this (and on the question of supersymmetry) can be found in [25].

[2]For the second factor $J = e^{iE_8}$: if $\partial Y \neq \emptyset$ then [14] J is not gauge-invariant, but is a section of a line bundle $\mathcal{L}^{-1}$ over the space of C-fields over $N = \partial Y$.

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over a twelve dimensional manifold Z. The full Chern-Simons coupling of M-theory is associated with I_{12}, which is well-defined and independent of the choice of Z and of the extension of G_4. The action can be written as

$$I_{12} = -\frac{1}{6}a^3\,, \tag{3.6}$$

where a is the cohomology class of $[\frac{G_4}{2\pi}]$. Witten [13, 14] have shown that there are two modifications to this: First that $a - \frac{\lambda}{2}$ is integral, and second we have to include $C_3 \wedge X_8$. Introduce an E_8 bundle V on Z whose characteristic class ω obeys $\omega = a - \lambda/2$. Witten have shown that

$$\frac{I_{12}}{2\pi} = \frac{i(E_8)}{2} + \frac{i(\text{R.S.})}{4} \tag{3.7}$$

and, including the above effects, the action takes the form

$$\frac{I_{12}}{2\pi} = -\frac{1}{6}\left(\omega - \frac{\lambda}{2}\right)\left[\left(\omega - \frac{\lambda}{2}\right)^2 - \frac{1}{8}(p_2 - \lambda^2)\right] \tag{3.8}$$

which is just $I_{\text{C.S.}}$ with the gravitational corrections turned on.

As for the Rarita-Schwinger path integral, $\frac{1}{2\pi}I_{12}$ can be half integral in general and has an anomaly that is cancelled from the one coming from the determinant of the Rarita-Schwinger operator $\det D_{rmR.S.}$. The combination shows up as

$$\det D_{\text{R.S.}} e^{iI_{\text{R.S.}}/2}\,. \tag{3.9}$$

The Rarita-Schwinger operator can be viewed as [1, 15] the Dirac operator coupled to $TX - 2\mathcal{O}$, since Y is a circle bundle over X, or equivelently, to $TZ - 4\mathcal{O}$ in twelve dimensions. Overall, one has the factor

$$\text{Pf}(D_{\text{R.S.}}) \exp\left(i\int_Z I_{12}\right). \tag{3.10}$$

Now $\text{Pf}(D_{\text{R.S.}})$ is a vector in a pfaffian line, so the above can be factorized into a modulus $|\text{Pf}(D_{\text{R.S.}})|$ and a phase $\Omega_M(C_3) \equiv (-1)^{I_{\text{R.S.}}/2}\exp\left(i\int_Z I_{12}\right)$.

Using the Atiyah-Patodi-Singer (APS) index theorem [26]–[28] one can relate the action to an index corrected by the reduced eta invariant $\overline{\eta} = \frac{h+\eta}{2}$, as

$$I(D) = \int_Z i_D - \overline{\eta} \tag{3.11}$$

so that the relevent integral in twelve dimensions can be written as

$$\int_Z \frac{I_{12}}{2\pi} = \frac{1}{2}I_{E_8} + \frac{1}{4}I_{\text{R.S.}} + \frac{h_{E_8} + \eta_{E_8}}{4} + \frac{h_{\text{R.S.}} + \eta_{\text{R.S.}}}{8} \tag{3.12}$$

Now the factor $(-1)^{I_{\text{R.S.}}}$ cancels the one coming from the index theorem, and taking into account the fact that the index is even, the phase derived in [1, 15] is

$$\Omega_M(C_3) = \exp\left[2\pi i\left(\frac{\overline{\eta}(D_{V(a)})}{2} + \frac{\overline{\eta}(D_{\text{R.S.}})}{4}\right)\right]. \tag{3.13}$$

4. Dimensional reduction from Y to X

In [1, 15], it was assumed that C_3 is a pullback from X. This implies that the topological invariant $\Omega_M(C_3)$ depends only on a and *not* on C_3, i.e. $\Omega_M(a)$. We would like to study the generalization of this to the case when the bundles in M-theory are not lifted from the type-IIA base, and so we consider the case of nontrivial Neveu-Schwarz H-field.

4.1 Reduction of the riemannian metric

The riemannian metric on the circle bundle Y is $g_Y = \pi^*(g_X) + \pi^*(e^{2\phi/3})\mathcal{A} \otimes \mathcal{A}$, where g_X is the riemannian metric on X, $e^{2\phi/3}$ is the norm of the Killing vector along S^1, (which, in this trivialization, is given by ∂_z) ϕ is the dilaton, i.e. a real function on X and $\mathcal{A}$ is a connection 1-form on the circle bundle Y. Note that the component of the curvature in the direction of the circle action,

$$R_{11} = e^{2\phi/3} = g_s^{2/3}. \tag{4.1}$$

Such a choice of riemannian metric is compatible with the principal bundle structure in the sense that the given circle action acts as isometries on Y.

Performing a rescaling to the above metric and using the identification (4.1), the desired metric ansatz for IIA is

$$g_Y = g_s^{4/3} g_{S^1} + t g_s^{-2/3} g_X \tag{4.2}$$

in the limit $t \to \infty$ *then* $g_s \to 0$.

4.2 Reduction of the differential forms G_4 and $G_7 = *G_4$

The reduction of the 4-form G_4 on Y gives rise to two differential forms on X, the Neveu-Schwarz 3-form H_3 and the Ramond-Ramond 4-form F_4. This is obtained as follows. For an oriented S^1 bundle with first Chern class $c_1(Y) = F_2 = d\mathcal{A} \in H^2(X, \mathbb{Z})$, we have a long exact sequence in cohomology called the *Gysin sequence* (cf. [29, Proposition (14.33)]).

$$\cdots \longrightarrow H^k(X,\mathbb{Z}) \xrightarrow{\pi^*} H^k(Y,\mathbb{Z}) \xrightarrow{\pi_*} H^{k-1}(X,\mathbb{Z}) \xrightarrow{F\cup} H^{k+1}(X,\mathbb{Z}) \longrightarrow \cdots \tag{4.3}$$

In particular, with $k = 4$, one sees that $F_2 \cup \pi_* G_4 = dF_4$, where F_4 is some differential 4-form on X. It follows that $d(\mathcal{A} \wedge \pi_* G_4 + F_4) = 0$. Therefore setting $H_3 = \pi_* G_4$, we see that H_3 is a closed form. Noting that $\pi_*(\mathcal{A}) = 1$, we arrive at the equation on Y (where it is understood that forms on X are pulled back to Y via π)

$$G_4 = F_4 + \mathcal{A} \wedge H_3\,. \tag{4.4}$$

Now the curvature 2-form $F_2 = d\mathcal{A}$ is basic, i.e., it is horizontal $i_v F_2 = 0$, and invariant $\mathcal{L}_v F_2 = 0$ where v is a vertical vector. In a local trivialization of the circle bundle where $\mathcal{A} = dz + \theta$ with θ being the connection on X, the above two conditions mean, respectively, that F_2 has no dz component and that it does not depend explicitly on z. Similarly, G_4 can be written in the given trivialization, as $G_4 = F_4 + dz \wedge H_3$.

Suppose that $G_4 \in \Omega^4(Y)$ and the curvature $F_2 \in \Omega^2(X)$ satisfies the Bianchi identities on Y that are given below, and which are obtained from the Euler-Lagrange equations for the Bosonic part of the action of eleven dimensional supergravity (cf. the formulae in equation (2.1)), namely $I_{\rm grav} + I_{G_4}$,[3]

$$dF_8 = H_3 \wedge F_6 \,, \tag{4.5}$$
$$dG_4 = 0 \,, \tag{4.6}$$
$$dG_7 = -\frac{1}{2} G_4 \wedge G_4 + X_8 \,. \tag{4.7}$$

where $F_8 = *_{10} F_2$, $F_6 = *_{10} F_4$, $G_7 = *_{11} G_4$ and X_8 is a basic differential form of degree 8 on Y, which is a Chern-Simons correction factor put in by hand. By applying the deRham differential on both sides of equation (4.7), we see that X_8 is a closed form.[4]

As argued above, the Bianchi identity $dG_4 = 0$ reduces to the Bianchi identities for the RR 4-form, the $NSNS$ 3-form and the RR 2-form field strengths, respectively,

$$dF_4 = H_3 \wedge F_2 \,, \tag{4.8}$$
$$dH_3 = 0 \,, \tag{4.9}$$
$$dF_2 = 0 \,. \tag{4.10}$$

From general principles, we can write $G_7 = H_7 + \mathcal{A} \wedge F_6$ where H_7 and F_6 are basic forms on Y - this is consistent with equation (4.4) since a standard computation shows that $i_v(*_{11} F_4) = *_{10} F_4$ and $i_v(*_{11}(A \wedge H_3)) = 0$. Then, using equation (4.7), one has

$$\begin{aligned} dG_7 &= dH_7 + F_2 \wedge F_6 - \mathcal{A} \wedge dF_6 \,, && (4.11)\\ &= -\frac{1}{2} F_4 \wedge F_4 - \mathcal{A} \wedge H_3 \wedge F_4 + X_8 \,, && (4.12) \end{aligned}$$

Eliminating dG_7 from the equations above, one arrives at

$$dH_7 = -F_2 \wedge F_6 + \mathcal{A} \wedge (dF_6 - H_3 \wedge F_4) - \frac{1}{2} F_4 \wedge F_4 + X_8 \,. \tag{4.13}$$

All of the terms in equation (4.13) are basic differential forms, with the sole exception of the term involving $\mathcal{A}$. Therefore contracting the terms of equation (4.13) with the vertical vector field v, and using the fact that $i_v(\mathcal{A}) = 1$ and $i_v(dF_6 - H_3 \wedge F_4) = 0$, we deduce the corresponding ten-dimensional Bianchi identities on X,

$$dF_8 = H_3 \wedge F_6 \,, \qquad dF_4 = H_3 \wedge F_2 \,, \tag{4.14}$$
$$dF_6 = H_3 \wedge F_4 \,, \qquad dH_3 = 0 \,, \qquad dF_2 = 0 \,, \tag{4.15}$$
$$dH_7 = -\frac{1}{2} F_4 \wedge F_4 - F_2 \wedge F_6 + X_8 \,. \tag{4.16}$$

[3]These equations admit solutions for a particular ansatz. For example, when $X_8 = 0$ and G_4 is proportional to a volume form of a four-dimensional factor in Y, this is the famous Freund-Rubin ansatz [32]. When G_4 is a flux through four-cycle(s) in Y, there are solutions with $X_8 \neq 0$, cf. [33] for different choices of Y.

[4]The one-loop coupling $\int C_3 \wedge X_8$ reduces to $\int B_2 \wedge X_8$ [35, 36].

Summarizing our discussion, from $G_4 \in \Omega^4(Y)$ satisfying the eleven dimensional Bianchi identities, we obtain $F = F_2 + F_4 + F_6 + F_8 \in \Omega^{\text{even}}(X)$ satisfying $(d - H_3\wedge)F = 0$, where we observe that[5] $F_0 = 0$ since H_3 is not exact and $F_8 \wedge H_3 = 0$ for dimension reasons. Therefore F determines a class in the twisted cohomology $H^{\text{even}}(X, H_3)$, where $H^\bullet(X, H_3)$ denotes the twisted cohomology, which is by definition the cohomology of the $\mathbb{Z}_2$-graded complex $(\Omega^\bullet(X), d - H_3\wedge)$, where the de Rham differential is replaced by $d - H_3\wedge$, cf. [45] (see also [46]). Our discussion in this section can also be summarized by the following diagram.

$$\begin{tikzcd} & G_4 \in H^4(Y,\mathbb{Z}) + (\text{Bianchi}) \arrow[dl] \arrow[dr] & \\ F \in H^{\text{even}}(X, H_3) & & H_3 \in H^3(X,\mathbb{Z}) \end{tikzcd} \tag{4.17}$$

4.3 Relating M-theory to K-theory

We are dealing with Dirac operators coupled to certain vector bundles. We are interested in the general case where the vector bundles are not lifted from the base. First we have the twisting by the tangent bundle, which leads to the Rarita-Schwinger operator. For this, one is dealing with natural bundles and so are lifted from the base. However, we also have the Dirac operator coupled to an E_8 vector bundle, which we would like to consider as not lifted from X. This leads to the appearance of eta-forms in the adiabatic limit of the reduced eta invariant of that Dirac operator.

4.3.1 Rarita-Schwinger operator

First we will look at $D_{\text{R.S.}}$. There are two contributions, one from h and the other from η. Recall that in dimensions $8n + 2$, $h_{D\otimes V_{\mathbb{R}}}$ is a topological invariant mod 2. For the contribution from h, the idea is to try to relate the spectrum on Y to that on X. The authors of [1, 15] choose functions Φ that transform as $\Phi \to e^{-ik\theta}\Phi$ under an S^1-rotation by an angle θ . The choice of functions depends on whether Y is compact or not. Correspondingly, in the compact case, one can choose the functions to be smooth $L^2(Y)$ wrt the metric that respects the circle bundle, and to be smooth in the noncompact case. In the case X and Y are compact, one can decompose the eta function as a sum over contribution from a given k. For $k = 0$, the phase is the same as the trivial circle bundle case, $i^{h^+_{\text{R.S.}}}$, with " + " referring to positive chirality, and for $k \neq 0$ there is no contribution from h.

The contribution from η is just the result of [1, 15], which is

$$\frac{\eta(s)}{2} = |R|^s \sum_{k=1}^{\infty} \left(a k^{-(s-1)} + b k^{-(s-3)} + c k^{-(s-5)} \right), \tag{4.18}$$

where the coefficients are given in terms of characteristic classes,

$$a = c_1(\mathcal{L}) \left(\text{rank}(V(a)) \hat{A}_8 - \lambda^2 \right) \tag{4.19}$$

[5]Hence note that, unlike the $[H_3] = 0$ case [2], we do not work in the massive type-IIA theory [34].

$$b = \frac{2}{9}\lambda c_1^3(\mathcal{L}) \tag{4.20}$$

$$c = 8\frac{c_1^5(\mathcal{L})}{5!}\,. \tag{4.21}$$

Then the above contributions combine as $\overline{\eta}_{\rm R.S.}$ and can be inserted in the phase (3.13).

4.3.2 E_8 Dirac operator

Now let us consider the E_8-coupled Dirac operator D on Y. Here we use the formalism of Bismut-Cheeger [37] (and Dai [38]) for calculating the adiabatic limit of the reduced eta invariant. Let $\mathcal{R}^X$ be the curvature of X and SX its spin bundle, with spin connection ∇^X induced from the Levi-Civita connection on X. Associated to the principal E_8 bundle P on X, we have a hermitean vector bundle $V(a)$ as in section 3, with a unitary connection $\nabla^{V(a)}$. Then the bundle $SX \otimes V(a)$ has a tensor product connection $\nabla = \nabla^X \otimes 1 + 1 \otimes \nabla^{V(a)}$. A natural representation of $Cl(X_p)$ on SX_p can be extended to a representation on $SX \otimes V(a)$.

Corresponding to the metric tg^X we have the Dirac operator $D^t_{V(a)}$, whose reduced eta invariant, when taken mod $\mathbb{Z}$, was shown by Bismut and Cheeger, and also by Dai, to be independent of t and has value of half the index of $D_{V(a)}$. When ${\rm Ker}D_{Y/X}$ is a vector bundle on X, one can use it to twist D_X. The connection on ${\rm Ker}D_{Y/X}$ is obtained as the projection of a unitary connection on the infinite-dimensional bundle $\mathcal{E} = L^2\left(\pi^{-1}(x), S_{\pi^{-1}(x)}\right)$ of smooth spinor sections along the S^1 fiber. For $x \in X$, have $E^x_p = \iota^*_x E_p \to \pi^{-1}(x) \cong S^1$ with $\pi^{-1}(x) \overset{\iota_x}{\hookrightarrow} Y$. This assumption that ${\rm Ker}D_{Y/X}$ is a vector bundle on X implies that there is no spectral flow for the Dirac operator on the fiber, i.e. the index of the Dirac operator on the mapping torus is zero. This means that there are no anomalies associated with that operator.

In this case, the adiabatic limit of the eta invariant on Y has a closed formula given by[6,7]

$$\lim_{t\to\infty} \overline{\eta}\left(D^t_{V(a)}\right) = \int_X \hat{A}\left(\mathcal{R}^X\right) \wedge \hat{\eta}_{V(a)} + \overline{\eta}\left(D_X \otimes {\rm ker}D_{Y/X}\right) + \frac{1}{2}h'\,, \tag{4.22}$$

where $\frac{1}{2}h'$ is a spin cobordism invariant, and where the eta-form is a differential form on X given by[8]

$$\hat{\eta}_{V(a)} = \frac{1}{\sqrt{\pi}} \int_0^\infty {\rm tr}^{\rm even}\left[\left(D_{Y/X} + \frac{c(T)}{4u}\right) e^{-(B_u)^2}\right] \frac{du}{2u^{\frac{1}{2}}}. \tag{4.23}$$

Here $B_u := \nabla^{V(a)} + u^{\frac{1}{2}} D_{Y/X} - \frac{c(T)}{4u^{\frac{1}{2}}}$ is the Bismut superconnection (see e.g. [39]), where c denotes Clifford multiplication and T is the torsion of the connection. The eta-form is of even degree, can be composed into homegenous even parts as $\hat{\eta} = \sum_{k=0}^{{\rm dim}X} \frac{1}{(2\pi i)^k}[\hat{\eta}]_{2k}$ and

[6]Bismut and Cheeger assumed D_{S^1} to be invertible, but Dai [38] has generalized the result by relaxing that condition.

[7]$\hat{\eta}$ that appears in this formula is renormalized by a factor of $(\frac{1}{2\pi i})^{[\frac{p+1}{2}]}$ for a p-form.

[8]This is analogous to the Heat Kernel representation of the eta invariant of D,

$$\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty {\rm tr}\left(De^{-uD^2}\right) \frac{du}{u^{\frac{1}{2}}}$$

has as the 0-form component the η-invariant of the Dirac operator along the fiber. This form arises as the spectral correction to the families version of the Atiyah-Patodi-Singer non-local elliptic boundary value problem. More precisely, recall that Z is a disk bundle over X, whose boundary is the circle bundle Y over X. The Bismut-Cheeger, Dai theorem in this context asserts that

$$\mathrm{ch}\left(\mathrm{Ind}(D_{Z/X})\right) = \int_{\mathbb{D}} \hat{A}(\mathcal{R}^{Z/X}) \wedge \hat{\eta}_{V(a)} + (\text{boundary correction})\,. \tag{4.24}$$

Here $\mathbb{D}$ is the disk which is the fiber of Z, $D_{Z/X}$ is the family of twisted Dirac operators along the fibers of Z that are parametrized by X with the Atiyah-Patodi-Singer boundary conditions and $\mathcal{R}^{Z/X}$ is the curvature of the vertical tangent bundle of Z. The last term in (4.24) is a correction term due to noninvertibility of the boundary operator.

The differential

$$d\hat{\eta} = \int_{S^1} \hat{A}\left(\mathcal{R}^{Y/X}\right) \wedge \mathrm{tr}\left(e^{-\frac{1}{2\pi i}\mathcal{R}^{V(a)}}\right) \tag{4.25}$$

is closed (not exact) and represents of the odd Chern class of $D_{Y/X}$. Here $\mathcal{R}^{Y/X}$ is the curvature of the connection on the vertical tangent bundle of Y obtained by projection of the Levi-Civita connection on Y, and $\mathcal{R}^{V(a)}$ is the curvature of the unitary connection on $V(a)$. After integrating over the fiber, we get an odd degree differential form on X. This formula in particular implies that the higher spectral flow vanishes.[9]

Even though we do not evaluate the above expression for the adiabatic limit, we point that we have reduced the adiabatic limit to an integral over the base, thus relating the M-theory data on the nontrivial circle bundle to the data of type IIA on X.

5. Type IIA theory

5.1 The partition function

Dimensional reduction of the eleven-dimensional action I_{11} on S^1 [41]–[43], with a radius R, leads to[10][11]

$$S_{IIA} = S_{NS} + S_{RR} + S_{\text{C.S.}} + S_{\text{Fermi}} + S_{\text{coupling}} \tag{5.1}$$

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int_X e^{-2\phi}\left[\mathcal{R} + 4d\phi\wedge *d\phi - \frac{1}{3}|H_3|^2\right]\, d\mathrm{vol} \tag{5.2}$$

$$S_{RR} = -\frac{1}{4\kappa_{10}^2}\int_X \left[|F_2|^2 + |F_4|^2\right]\, d\mathrm{vol} \tag{5.3}$$

$$S_{\text{C.S.}} = -\frac{1}{4\kappa_{10}^2}\int_X B_2\wedge F_4\wedge F_4 \tag{5.4}$$

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[9] If Y has a nonempty boundary then [40] the results of Bismut and Cheeger still hold provided one keeps the invertibility condition.

[10] The S_{coupling} is the supersymmetric completion of the action by algebraic terms in various fields, e.g. bilinear and quartic in the fermions as well as coupling of the bilinear terms to the p-forms.

[11] Up to total derivative one can rewrite the Chern-Simons term in terms of H_3 rather than B_2, namely $\int_X H_3\wedge C_3\wedge F_4$.

$$S_{\rm Fermi} = \frac{i}{2}\int_X \left[\bar\psi D_{\rm R.S.}\psi + \bar\lambda D\lambda\right]\, d{\rm vol}\,, \tag{5.5}$$

where $\mathcal{R}$ is the scalar curvature of X, $F_4 = G_4 - \mathcal{A}\wedge H_3$ is the gauge-invariant RR 4-form field strength, and D is the Dirac operator acting on the dilatino λ, the superpartner of the dilaton ϕ. The gravitational coupling constant in eleven dimensions, κ_{11}, is related[12] to the one in ten dimensions by $\kappa_{10}^2 = \frac{\kappa_{11}^2}{2\pi R}$. As in the case for M-theory, $S_{\rm coupling}$ involves coupling of the fermions to the forms, as well as self-couplings, and we will not use this in this paper.

The partition function of type IIA string theory is of the form

$$Z_{\rm IIA} \sim Z_{\rm NS} Z_{\rm RR} Z_{\rm C.S.} Z_{\rm Fermi} Z_{\rm coupling}\,. \tag{5.6}$$

The Ramond-Ramond part is encoded in the theta function Θ_{IIA} coming from summing over the RR forms [1, 15]. In [2] Z_{NS} for flat potentials namely

$$\exp\left[\pi i \int_X B_2^{(0)} F_4 F_4\right] \tag{5.7}$$

as well as $Z_{\rm Fermi}$, together with 1-loop determinants were considered. In addition, the authors also include contribution to the partition function from 1-loop corrections $\int_X B_2^{(0)} X_8$ to the effective action and the effect of membrane instantons. Here we focus on the Ramond-Ramond part and study the generalization to the case $[H_3] \neq 0$.

5.2 Twisted K-theory

Our goal in this section is summarized in the following diagram.

$$\begin{tikzcd} & E_8 \text{ bundle over } Y + \text{(Bianchi)} \arrow[dl] \arrow[dr] & \\ F \in K^0(X,H) & & LE_8 \text{ bundle over } X \end{tikzcd} \tag{5.8}$$

This is the analog of what was described in section 4.2.

It has been suggested in [44] that the E_8 bundle in M-theory can be related to an LE_8 bundle in type IIA (on X). Starting from principal E_8 bundle over Y, the dimensional reduction of the M-theory to type IIA gives a LE_8 bundle P' in ten dimenions, characterized by the 3-form $H_3 = \int_{S^1} G_4$ (or equivalently $H_3 = \iota_v G_4$).

$$\left\{\begin{array}{ccc} E_8 & \to & P \\ & & \downarrow \\ S^1 & \to & Y \\ & & \downarrow \\ & & X \end{array}\right\} \longrightarrow \left\{\begin{array}{ccc} LE_8 & \to & Q \\ & & \downarrow \\ & & X \end{array}\right\} \tag{5.9}$$

Note that since E_8 is an approximate $K(\mathbb{Z},3)$ up to dimension 14, it follows that principal E_8 bundles over Y are classified by $H^4(Y,\mathbb{Z})$. More precisely, the characteristic class of the

[12]Explicitly $\kappa_{11}^2 = 2\pi R\kappa_{10}^2$ and $R = g_s\sqrt{\alpha'}$ and $\kappa_{11}^2 = \frac{1}{2}(2\pi)^8 g_s^3 \alpha'^{\frac{9}{2}}$ give $\kappa_{10}^2 = \frac{1}{2}(2\pi)^7 g_s^2 \alpha'^4$.

E_8 bundle is the restriction of the first Pontjagin class p_1 to the 4-spheres in the 4-skeleton of the base manifold, $G_4 = \lambda(p_1) \in H^4(Y, \mathbb{Z})$. Then the class on LE_8 is $\pi_* \lambda(p_1) \in H^3(X, \mathbb{Z})$. There exists a LE_8 bundle, unique up to isomorphism, such that the Dixmier-Douady class $DD(LE_8) = \pi_* \lambda(p_1)$. For $m \in X$, $\pi^{-1}(m) = S^1$, one has

$$C^\infty \left(\pi^{-1}(m), P|_{\pi^{-1}(m)}\right) \cong LE_8 \,. \tag{5.10}$$

This gives the fibration above with $Q = \bigcup_{m \in X} C^\infty \left(\pi^{-1}(m), P|_{\pi^{-1}(m)}\right)$ so $DD(Q) = \pi_* \lambda(p_1) = H_3$. The obstruction to lifting the LE_8 bundle Q to an $\widehat{LE_8}$ bundle P', covering Q, is the Dixmier-Douady class.

$$\begin{array}{ccc} \widehat{LE_8} & \to & P' \\ & & \downarrow \\ & & X \end{array} \tag{5.11}$$

That is, such a lift is possible only when $H_3 = dB_2$. Therefore, in the presence of F_0, only the trivial case (in the sense of the $NSNS$ 3-form) can be seen in loop group picture.

We have seen in section 4.2 that we can derive from the 4-form G_4 on Y, a 4-form F_4 on X. Recalling that $F_2 = d\mathcal{A}$ and considering the inhomogeneous even degree form[13] $F = F_2 + F_4 + F_6 + F_8$, where $F_8 = *_{10} F_2$ and F_6 is obtained by dimensional reduction of $G_7 = *G_4$ as in section 4.2, we have seen that F is $d - H$ closed. Then using the fact that the twisted Chern character $ch_H : K^0(X, H) \to H^{\text{even}}(X, H)$ is an isomorphism over the reals, we obtain an element $F \in K^0(X, H) \otimes \mathbb{R}$. Unfortunately, we do not know at this time how to lift this to a class in $K^0(X, H)$, as methods used when $H_3 = 0$ do not seem to apply in the twisted case. We leave this as an open problem.

5.3 Twisted K-theory torus and theta functions

In the presence of branes and and and H-flux, the RR fields F are determined by the twisted K-theory classes $x \in K(X, H)$ via the twisted Chern map [4, 5, 7, 45, 47]

$$\frac{F(x)}{2\pi} = ch_H(x) \sqrt{\hat{A}(X)} \in H^\bullet(X, H) \,, \tag{5.12}$$

where $\widehat{A}$ is the A-roof genus.

It turns out that the conjugate of x, $\bar{x} \in K(X, -H)$

$$\frac{F(\bar{x})}{2\pi} = ch_{-H}(\bar{x}) \sqrt{\hat{A}(X)} \in H^\bullet(X, -H) \,. \tag{5.13}$$

Setting $F = \sum_{n=1}^4 F_{2n}$ for the gauge-invariant field strengths, the RR field EOM can be written succinctly as[14]

$$dF = H_3 \wedge F \,. \tag{5.14}$$

[13] Since $F_0 = 0$ then there is no F_{10}.

[14] In order to make the RR field strengths homogeneous of degree zero, one can [52] use K-theory with coefficients in $K(pt) \otimes \mathbb{R} \cong \mathbb{R}[[u, u^{-1}]]$ where the inverse Bott element $u \in K^2(pt)$ has degree 2, and look at

Putting it another way, the RR field EOM on the level of differential forms says that the RR fields determine elements in twisted cohomology, $H^\bullet(X,H)$. At the level of cohomology this implies $H_3 \wedge F_n = 0$. In K_H, or more precisely in the Atiyah-Hirzebruch spectral sequence ($AHSS$), this becomes[15]

$$(H + Sq^3) \cup F_n = 0 \tag{5.15}$$

[15] argue (and conjecture) that the M-theory partition function on a circle bundle can be written in terms of fields satisfying (5.15).

A special case of the cup product pairing in twisted K-theory followed by the standard index pairing of elements of K-theory with the Dirac operator, explains the upper horizontal arrows in the diagram,

$$\begin{array}{ccccc} K^\bullet(X,H)\times K^\bullet(X,-H) & \longrightarrow & K^0(X) & \xrightarrow{\text{index}} & \mathbb{Z} \\ {\scriptstyle ch_H\times ch_{-H}}\big\downarrow & & \big\downarrow{\scriptstyle ch} & & \big\downarrow\| \\ H^\bullet(X,H)\times H^\bullet(X,-H) & \longrightarrow & H^{\text{even}}(X) & \xrightarrow{\int_X \widehat{A}(X)\wedge} & \mathbb{Z} \end{array} \tag{5.16}$$

The bottom horizontal arrows are cup product in twisted cohomology followed by cup product by $\widehat{A}(X)$ and by integration. By the Atiyah-Singer index theorem, the diagram (5.16) commutes. Therefore the normalization given to the Chern character in the definition of $\frac{F(x)}{2\pi}$ makes the pairings in twisted K-theory and twisted cohomology isometric.

As noted by Witten [48], there is a subtlety in the self-duality $*F = F$. It is in fact not possible to impose a classical quantization law on the periods of a self-dual p-form. This is because one cannot simultaneously measure anticommuting periods, i.e. ones whose intersection number is non-zero. The way around this is [48, 7] to interpret this self-duality as a statement in the quantum theory and sum over half the fluxes, i.e. over a maximal set of commuting periods. So we need a phase space (in twisted K-theory) with a polarization or lagrangian subspace that naturally splits the forms in half. The lattice is $\Gamma_{K_H} = K(X,H)/K(X,H)_{tors}$. This is isomorphic to the image of the modified chern character homomorphism of $\mathbb{Z}_2$-graded rings,

$$\sqrt{\hat{A}(X)} \wedge ch_H : K(X,H) \to H^{\text{even}}(X,H;\mathbb{R}) \tag{5.17}$$

and the kernel is $K(X,H)_{tors}$, the torsion subgroup. The lattice is unimodular by Poincaré duality in twisted K-theory. In what follows, we give an analog of some of the constructions given in [1] and [2].

the corresponding chern character as a homomorphism of $\mathbb{Z}$-graded rings,

$$ch : K(X,H) \to H^{\text{even}}\left(X,H;\mathbb{R}\left[[u,u^{-1}]\right]\right).$$

Then the total RR field strength is written as

$$F = F_0 + u^{-1}F_2 + u^{-2}F_4 + u^{-3}F_6 + u^{-4}F_8 + u^{-5}F_{10}.$$

[15] There has also been proposals for S-duality-covariant extensions of AHSS in [49, 50].

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First, using equations (5.12) and (5.13), we get the metric (that gives the kinetic energy)

$$g(x,y) = \frac{1}{2\pi^2}\int_X F(x) \wedge *F(y) \tag{5.18}$$

which is defined on the lattice Γ_{K_H}, and which determines a homogeneous metric on the twisted K-theory torus $\mathcal{T}_H(X) = (K(X,H) \otimes \mathbb{R})/\Gamma_{K_H}$.

Similarly consider the bilinear form on the lattice Γ_{K_H}

$$\omega(x,y) = \frac{1}{2\pi^2}\int_X F(x) \wedge F(\bar{y}) = I(x \otimes \bar{y})\,, \qquad \forall\, x, y \in K(X,H)\,, \tag{5.19}$$

where we notice that $x \otimes \bar{y} \in K(X)$. Here

$$I(\xi) = \int_X \hat{A}(X) \wedge ch(\xi)\,, \qquad \forall\, \xi \in K(X)\,. \tag{5.20}$$

For a torsion class $x_0 \in K(X,H)_{tors}$ and for any $x \in K(X,H)$, have $\omega(x,x_0) = 0 = g(x,x_0)$ since $nx_0 = 0$. This implies that $\omega(\ ,\ )$ and $g(\ ,\ )$ are well-defined on the lattice $\Gamma_{K_H} = K(X,H)/K(X,H)_{\mathrm{tors}}$. If X is a spin manifold and $\dim(X) = 4n+2$, then ω is antisymmetric: this essentially uses the arguments of [48] for the untwisted case. The only terms in $ch(\xi)$ that contribute to the value of $I(\xi)$ are terms of degree $4k+2$ where $k \leq n$, since $\hat{A}(X)$ has components only of degree $4l$ for some l. That is, the only terms in $ch(\xi)$ that contribute to the value of $I(\xi)$ are $ch_j(\xi)$ for j odd. These terms are odd under the transformation $\xi \to \bar{\xi}$, since $ch_j(\bar{\xi}) = -ch_j(\xi)$ for j odd, so that $I(\xi) = -I(\bar{\xi})$. This implies $\omega(x,y) = I(x \otimes \bar{y}) = -I(y \otimes \bar{x}) = -\omega(y,x)$ is antisymmetric. Then ω determines a homogeneous differential 2-form on the twisted K-theory torus $\mathcal{T}_H(X)$ which is closed and integral.

The form ω is unimodular (i.e. $\frac{1}{2\pi i}\omega$ is integral and $\int_{\mathcal{T}_H(X)} e^{\frac{1}{2\pi i}\omega} = 1$) on the lattice Γ_{K_H} because X is spin and because of Poincaré duality in twisted K-theory [9], i.e. the top line in (5.16) is a unimodular pairing, and the lattice Γ_{K_H} is symplectic.

The pair (g,ω) determine a Kähler form that is an integral form. By the Kodaira embedding theorem,[16] the twisted K-theory torus $\mathcal{T}_H(X)$ is a smooth projective algebraic variety. Since the lattice Γ_{K_H} is symplectic, it has a lagrangian decomposition $\Gamma_1 \oplus \Gamma_2$ (or polarization) in terms of commutative sublattices Γ_1 and Γ_2. Because of the duality between Γ_1 and Γ_2, there is an element $\theta_K \in \Gamma_1$ as argued in [1], [2] so that, for any $y \in \Gamma_2$, $\Omega(y) = (-1)^{(\theta_K, y)}$. Summing over half the fluxes now amounts to summing over the lagrangian sublattice Γ_1, which gives the type-IIA partition function,

$$\Theta_{IIA}(H : \tilde{\tau}) = e^{iu}\sum_{x \in \Gamma_1} e^{i\pi\tilde{\tau}_K(x + \frac{1}{2}\theta_K)}\Omega(x)\,, \tag{5.21}$$

where $\tilde{\tau}$ are the period matrices when $F(x)$ is twisted. This is a theta function on the torus $\mathcal{T}_H(X)$. Explicitly, the quadratic form on $\Gamma_1 \otimes \mathbb{R}$ is determined by:

$$\mathrm{Re}\,\tilde{\tau}_K\left(x + \frac{1}{2}\theta_K\right) = \frac{1}{(2\pi)^2}\int_X (-F_2 \wedge F_8 + F_4 \wedge F_6) \tag{5.22}$$

[16] Which says that a compact complex manifold which admits a positive line bundle can be holomorphically embedded in complex projective space.

$$\operatorname{Im}\tilde{\tau}_K\left(x+\frac{1}{2}\theta_K\right) = \frac{1}{(2\pi)^2}\int_X (F_2\wedge *F_2 + F_4\wedge *F_4)\,, \tag{5.23}$$

where $\operatorname{Im}\tilde{\tau}_K > 0$, and u is given by

$$u = -\pi\operatorname{Re}\tilde{\tau}_K\left(\frac{1}{2}\theta_K\right). \tag{5.24}$$

As in [1, 2], the function $\Omega(x)$ given in equation (5.21) satisfies the identity

$$\Omega(x+y) = \Omega(x)\Omega(y)(-1)^{\omega(x,y)}\,. \tag{5.25}$$

There are potentially many such functions, but we will make a particular choice as suggested in [48]. Let $q(V)$ denote the parity of the (real) dimension of the space of chiral zero modes of the real Dirac operator coupled to the real vector bundle V, D_V. It is a topological invariant in $8n+2$ dimensions, and in particular for X. Define $\Omega(x) = (-1)^{q(x\otimes\bar{x})}$, where we observe that $x\otimes\bar{x}\in KO(X)$ for all $x\in K(X,H)$. Assuming Ω to be identically one when restricted to $K(X,H)_{\rm tors}$, it can then be regarded as a function on the lattice Γ_{K_H}. By the argument in [48], it determines a holomorphic and hermitean line bundle $\mathcal{L}$ over $\mathcal{T}_H(X)$, with a connection having curvature equal to ω. $\mathcal{L}$ has a holomorphic section Θ as defined above which is unique (up to multiplication by scalars) by the Riemann-Roch theorem which says in this case that $\dim H^0(\mathcal{T}_H(X),\mathcal{L}) = \int_{\mathcal{T}_H(X)} e^{c_1(\mathcal{L})} = 1$, since $c_1(\mathcal{L}) = \frac{1}{2\pi i}\omega$ and ω is unimodular. Notice that if we changed the spin structure on X, then the twisted K-theory torus $\mathcal{T}_H(X)$ doesn't change, but what changes is the choice of section of the line bundle $\mathcal{L}$ over $\mathcal{T}_H(X)$, i.e. the theta function changes, since Ω depends on the choice of spin structure. In the case when $X = W\times\Sigma$ where W is a compact spin 8 dimensional manifold and Σ is a compact Riemann surface, then the (mod 2) index is often a nontrivial function on the subspace of spin structures arising from Σ.

For the argument above to work, we need to know that the function Ω is constant on the torsion subgroup $K(X,H)_{tors}$. This may not be satisfied in general, and so may give rise to an anomaly. However, in an analogous setting, Hopkins and Singer [51] remove this anomaly by a deeper analysis of the situation.

To connect with the dimensional reduction metric ansatz, it is convenient, as in [1, 2], to choose a polarization that keeps only positive powers of t in the kinetic term upon scaling $g_X \to tg_X$ under which $\int_X *1|F_{2p}|^2 \to t^{(5-2p)}\int_X *1|F_{2p}|^2$. Since coefficients of $|F_{2p}|^2$ for $p\geq 3$ tend to zero in the limit $t\to\infty$, we keep positive powers in the expansion

$$\sum_{p=1}^{4} t^{(5-2p)}|F_{2p}|^2\,. \tag{5.26}$$

The correction due to lagrangian or polarization amounts to shifting the class $x\in\Gamma_1$ by the theta element $\frac{1}{2}\theta_K$,

$$\left[\frac{F(x)}{2\pi}\right] = ch_H\left(x+\frac{1}{2}\theta_K\right)\sqrt{\hat{A}(X)}\,. \tag{5.27}$$

To get only positive powers in t, we take Γ_1 to be the complementary lattice to the lattice Γ_2 that consists of K-theory classes x such that $ch_n(x) = 0$ for $n = 0,1,2$. The dominant contribution comes from the K-theory class $x \in \Gamma_1$ with $F_2(x) = 0$ (still assuming $F_0 = 0$). The sublattice of such classes with virtual dimensions 0 such that $c_1(x) = 0$ is Γ_1'. Then the IIA partition function reduces to

$$\Theta_{\mathrm{IIA}}(H : \tilde{\tau}) = e^{iu} \sum_{x \in \Gamma_1'} e^{-\pi t \int_X ||F_4||^2} e^{i\pi \int_X F_4 \wedge F_6} \Omega(x) . \tag{5.28}$$

The corresponding modification to the phase $\mathrm{Im}(S_{IIA}) = -2\pi\tilde{\Phi}$ of the action (5.1) is found by expanding $F_4 F_6$ and picking the appropriate degree. We note that many of the computations in [2] that were performed for the cohomologically trivial case, extends to the nontrivial case as well. For example,

$$\tilde{\Phi} = \Phi + \frac{1}{8\pi^2}\left[B_2 F_4^2 + B_2 F_2 F_6 + \frac{1}{3} B_2^3 F_2^2\right] , \tag{5.29}$$

where Φ is the result of [1], still holds, eventhough it was derived when $H_3 = dB_2$. Basically, this is due to the fact that by lifting to the total space of the LE_8 bundle, H_3 becomes dB_2, by the general property of characteristic classes (recall that H_3 is the Dixmier-Douady characteristic class). One might be interested in setting $F_2 = 0$ to simplify the above expression, in which case, the contribution is just the one coming from the Chern-Simons part, $S_{\mathrm{C.S.}}$, of the type-IIA action.

Consider a PU bundle E over X, with 3-curvature H, cf. [45]. Then any other 3-curvature H' for E, has the property that $H' = H + dB$ where B is a global 2-form on X, so that $K(X, H) \cong K(X, H + dB)$. Also if E' is another PU bundle over X with 3-curvature H'', then again we have that $H'' = H + dB'$, where B' is a global 2-form on X, so that $K(X, H) \cong K(X, H + dB')$, cf. [45]. In particular, this implies that the twisted K-theory tori are isomorphic,

$$\mathcal{T}_H(X) \cong \mathcal{T}_{H+dB}(X) . \tag{5.30}$$

A gauge transformation for B, $B_2 \to B_2 + f_2$, $f_2 \in H^2(X, \mathbb{Z})$ will leave F invariant if this gauge transformation also acts on $K(X, H)$ as $x \to \pi^* \mathcal{L}(-f_2) \otimes \tilde{x}$, $\tilde{x} \in K(X, H)$, where the line bundle $\mathcal{L}(-f_2)$ has a Chern class given by $c_1(\mathcal{L}(-f_2)) = -f_2$. Then this gauge transformation acts as an automorphism of Γ_K, preserving the symplectic form ω on the twisted K-theory torus.

6. Discussion

In this paper we considered relating the fields of M-theory to those of type IIA in the large volume limit. We derived the RR fields of type IIA from the M-theory 4-form G_4 satisfying the Bianchi equations, and have shown that those fields define elements in twisted cohomology $H^{\mathrm{even}}(X, H_3)$. In order to write the partition function of type IIA , we constructed the twisted K-theory torus. We also have considered the topological part of the M-theory partition function, and, for the general case when the E_8 vector bundle that twists the Dirac operator, is not lifted from the base, and the NS field H_3 is nontrivial in cohomology, we have written the eta invariant (that determines the phase) in the adiabatic limit as an integral in type IIA . We have also discussed the similarities and differences with the cases $B = 0$ [1] and $H = dB$ [2], considered before.

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In relating the eta invariant in M-theory (i.e. on Y) to an integral in type IIA (on X), we naturally encountered the eta-forms in the integrand. It would be very interesting to compute the eta-forms for nontrivial circle bundles Y over X for the case when the Dirac operator on the total space is coupled to vector bundles that are not lifted from the base. Zhang [53] has computed $\hat{\eta}$ in the lifted case. It was also computed by Goette [54] using G-equivariant eta invariants for the nontrivial circle bundle. It might also be interesting to give a physical interpretation to the various form components of $\hat{\eta}$ that show up in the phase, perhaps in analogy to Witten's global anomaly for the degree zero component.

T-duality [55, 56] relates type IIA to type IIB string theory and can be implemented at the level of the effective action [57], and K-theory [58, 59]. The RR fields of type IIB, in the presence of the $NSNS$ field H_3 and in the absence of branes, are determined by an element $\hat{x} \in K^1_H(X)$. Since $K^1_H(X) \cong \widetilde{K}^0_H(S^1 \times X)$, one can view [7] $ch_H(x)$ as an even class on $S^1 \times X$, which upon integration over S^1 gives an odd element $i_* ch_H(x)$. This is what one expects from T-duality for the case X is the total space of a circle bundle. Then T-duality maps the 4-form G_4 of type IIA to the self-dual 5-form F_5 of type IIB. It is then resonable to believe that one can relate properties of one to those of the other. In fact, as explained by Witten [48], it is possible to deduce the integrality conditions on G_4 from the quantum mechanics of the self-dual five-form field strength. It would be interesting to implement T-duality of IIA/IIB at the level of K-theory theta functions, and deduce the T-duality anomalies, along the lines of [2], from the IIB picture, or alternatively, follow the methods of [51].

We have constrained our discussion mostly to the RR part of the partition function (twisted by H_3). It would be interesting to include the other parts of the partition function (fermions, one-loop and quantum corrections) and find the corresponding (T-duality) anomaly-free partition function. In order to get a T-duality anomaly-free partition functions, [2] considered super-theta functions by making the K-theory torus into a supertorus. It would be interesting to describe such functions in this context.

In trying to derive the answer from the LE_8 picture, one can give a description of the twisting field H_3 and the RR 4-form field that is being twisted, but only in twisted cohomology. It would be interesting to see how lifting problem might be solved in the twisted case, and give the corresponding description in twisted K-theory (see subsection 5.2).

In order to be able to define the theta functions, one needed the condition on the $\mathbb{Z}_2$-valued function Ω to be identically one on $K(X,H)_{\text{torsion}}$, in order to descend to a function on Γ_{K_H} and thus define $\mathcal{L}$ and the theta functions. In [1], this condition was shown to be equivalent to a condition on the Stiefel-Whitney class $W_7(X) = 0$. If $\Omega \neq 1$ then there is a possible anomaly. It would be interesting to see, as suggested from the work of Hopkins and Singer [51], how a refinement of the construction shows the absence of such an anomaly, in the presence of the twisting by H_3.

The anomaly cancellation condition, in the untwisted case, $Sq^3 = 0$, shows up as an obstruction to lifting cohomology to K-theory and also as a condition for modding out by torsion in the M-theory phase (averaging over the torsion classes) as in [1, 2]. It would be interesting to see if the corresponding condition in the twisted case [18, 19, 7] $Sq^3 + [H] = 0$

can be viewed as an obstruction to lifting to twisted K-theory (beyond AHSS), and as the condition for the M-theory phase, and also perhaps as an obstruction to creating LE_8 bundles. Obviously, a lot of work needs to be done, and we hope to address some of those interesting problems in the future.

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